Reducing Strangeness-Free Delay Differential-Algebraic Systems to Neutral Systems with Application to $\mathcal{H}_2$ Analysis

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Abstract

The paper is devoted to the reduction of linear time-invariant delay differential-algebraic systems (singular systems) to neutral type non-singular systems. We consider the class of strangeness-free systems, which is broader than the commonly investigated class of regular impulse-free systems. Strangeness-free systems have some important properties, like the existence and the unicity of the solution, and the standard relation between the spectrum and the stability property of the system. The contribution is twofold. First, we show how to reduce a strangeness-free control system to a neutral type non-singular one by adding low rank term to the system. The second contribution is in applying the result to the stability analysis and the $\mathcal{H}_2$ norm characterization for the differential-algebraic systems.

Keywords : Delay differential-algebraic equations, norm of the transfer matrix, neutral equations, strangeness-free systems.
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Abstract

The paper is devoted to the reduction of linear time-invariant delay differential-algebraic systems (singular systems) to neutral type non-singular systems. We consider the class of strangeness-free systems, which is broader than the commonly investigated class of regular impulse-free systems. Strangeness-free systems have some important properties, like the existence and the unicity of the solution, and the standard relation between the spectrum and the stability property of the system. The contribution is twofold. First, we show how to reduce a strangeness-free control system to a neutral type non-singular one by adding low rank term to the system. The second contribution is in applying the result to the stability analysis and the $\mathcal{H}_2$ norm characterization for the differential-algebraic systems.

1 Introduction

The paper is devoted to the analysis of linear time-invariant systems of delay differential-algebraic equations (DDAEs), which are also called singular delay systems.

In the delay-free case the state variable of the system of DDAEs is constrained by differential and algebraic equations, whereas in the time-delay case the situation is more complex, as the delay differential-algebraic system consists, generally speaking, not only of deferential and algebraic equations, but also of the delay difference equations. There are many applications of such systems, as, in particular, the class of linear systems of DDAEs includes systems with a dynamic feedback controller, systems with the delayed measurements, systems with the delayed input, neutral-type systems, and some other classes of systems.

In contrast to differential-difference systems, for some classes of linear time-invariant systems of DDAEs even some basic properties can not be guaranteed. The properties like continuability and uniqueness of the solutions, the connection between the exponential stability and the location of the spectrum on the complex plain do not hold, generally speaking. Many papers on DDAEs are devoted to regular impulse-free systems (in this paper, we call them systems with strangeness-free non-delayed part) – a class, where one can guarantee all the listed properties; see, e. g., [21, 6, 20, 8]. In this paper, we consider the class of strangeness-free systems, which is broader. This concept comes from the theory of delay-free differential-algebraic time-varying systems. It has been adapted for time-delay systems in [4]. Linear strangeness-free systems can be brought by elementary row operations into "stepped" form, consisting of three parts: differential-difference, difference and algebraic. In [4] the authors show, how to reduce the system of DDAEs to the system of NDDEs by differentiation and time-shift operation. The differentiation obviously introduce additional dynamics into the system, but, as has been shown in [4], such dynamics does not break the stability under some natural constraints on the initial states. The approach has been applied to prove the fundamental property that the exponential stability of systems of strangeness-free DDAEs is equivalent to the negativity of the spectral abscissa.

Our contribution is twofold. First contribution is the algorithm that allows to reduce any strangeness-free system to a neutral type non-singular system. We divide the process into two steps. On the first step we reduce a strangeness-free system to a system with strangeness-free non-delayed part, which is equivalent to the original one in the frequency-domain, as both their spectrum and their transfer matrices coincide. On the second step we reduce the obtained system to the non-singular neutral type systems. The transfer matrices of the original system and the corresponding non-singular system are equal, but these systems are not strictly equivalent, as the spectrum of the latter contains additional modes, which can be prescribed. Nevertheless, if we choose additional modes to be stable with a sufficiently large stability margin, such transformation preserves exponential stability, and the decay rate of the solution that is helpful for the stability analysis of differential-algebraic systems via the stability analysis of non-singular systems of NDDEs that are better investigated. See, to name a few, [10, 19, 15, 18, 7] for the stability analysis of neutral type systems, [9, 13, 12, 2] for the construction of the exponential estimates for the solutions. Thus, we develop the idea of reduction of DDAEs to NDDEs, introduced in [4]. As an important distinction from existing works, our approach does not rely on a preliminary transformation of the system to the "stepped" form, and leads to the neutral type systems with the prescribed additional dynamics. The resulting system is expressed in terms of the original system matrices, and both systems are very similar up to the addition of low rank terms.

The second main contribution is the applying of the proposed transformation to the characterizing of the $\mathcal{H}_2$ norm of the transfer function for systems of DDAEs. To the best of our knowledge, this task have never been addressed in the literature for systems of DDAEs, but it has already been addressed for neutral type systems [11]. The problem is that for differential-algebraic systems the $\mathcal{H}_2$ norm is not necessary finite. In this paper, we give an algebraic necessary and sufficient condition for the finiteness of the norm, and point out some sufficient conditions, when it can be computed for the system of DDAEs via the corresponding system of NDDEs.

The organization of the paper is as follows. We introduce basic definitions and some auxiliary results in Section 2. Section 3 is devoted to the reduction of strangeness-free systems to the systems with strangeness-free non-delayed part. Section 4 is divided into three subsections: some auxiliary elements are introduced in the first subsection, the algorithm for reducing of systems with strangeness-free non-delayed part to the non-singular neutral type systems is given in Subsection 4.2, the application to the stability analysis is discussed in Subsection 4.3. In Section 5, we show how to apply the obtained results to the $\mathcal{H}_2$ norm characterization and computation. In particular, an algebraic criterion of finiteness of the $\mathcal{H}_2$ norm is given in Subsection 5.1, and an approach for computation of the $\mathcal{H}_2$ norm in some particular cases is presented in Subsection 5.2. Two illustrative examples and some concluding remarks end the paper.

Throughout the paper, by I and 0 we denote identity and null matrices of appropriate dimension, which is clear from the context, $\mathbf{R}^{n\times m}$ is the space of real matrices $n\times m$, $\mathbf{C}$ is the space of complex numbers.

2 Definitions and basic properties

We consider the following linear time-invariant delay system:

$$\frac{d}{dt}\big(Ex(t)\big) = A_0x(t) + A_1x(t-h) + Bu(t) \tag{1}$$

with the output

$$y(t) = Cx(t),$$

where

$$E, A_0, A_1 \in \mathbf{R}^{n\times n},\ B \in \mathbf{R}^{n\times p},\ C \in \mathbf{R}^{v\times n},$$

$x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^p$ is the input, $y(t) \in \mathbf{R}^v$, $h > 0$ is the constant delay. We focus on the case of singular matrix E.

For a fixed input $u(t)$, $t \in [0,\infty)$, which is assumed to be piecewise-continuous (piecewise continuously differentiable, if the system contains the derivative of $u(t)$), a piecewise-continuous function $x(t,\varphi)$, $t \in [-h,\infty)$, such that $Ex(t,\varphi)$ is absolutely continuous on $[0,\infty)$, is called the *solution* of system (1), if it satisfies (1) on $[0,\infty)$, and

$$x(\theta,\varphi) = \varphi(\theta), \quad \theta \in [-h,0],$$

where $\varphi(\theta)$, $\theta \in [-h, 0]$, is a consistent initial function.

Let

$$H(s) = sE - A_0 - e^{-sh} A_1$$

be the *characteristic matrix* for system (1),

$$\Lambda = \{ s \in \mathbf{C} \mid \det H(s) = 0 \}$$

be the *spectrum*,

$$G(s) = CH^{-1}(s)B$$

be the *transfer matrix* of this system.

Definition 1. We say that system (1) is *exponentially stable*, if there exist $\gamma > 0$ and $\sigma > 0$ such that any solution of the system with $u \equiv 0$ satisfies the inequality

$$\|x(t, \varphi)\| \leqslant \gamma e^{-\sigma t} \sup_{\theta \in [-h, 0]} \|\varphi(\theta)\|.$$

Generally speaking, there is no direct connection between the spectrum location and the stability of system (1) (see, Example 1.2 in [4]). Sometimes systems with singular E have "strange" properties. To illustrate this, we give two simple examples.

Example 1. *We consider the system*

$$\begin{aligned} \dot{x}_1(t) &= 2x_1(t) - x_3(t), \\ 0 &= x_2(t) + x_1(t-h) - x_3(t-h) + u(t), \\ 0 &= x_2(t) + x_1(t-h) - x_3(t-h) + 2u(t). \end{aligned}$$

It is easy to see that there exist infinitely many solutions, if $u \equiv 0$. In fact, $x_3(t)$ is arbitrary in this case. But if $u \not\equiv 0$, the system has no solutions.

Note also that the characteristic function is

$$f(s) = \det H(s) \equiv 0$$

that means that the spectrum of the system coincides with the whole complex plain.

Example 2. *It is easy to note that the system*

$$\begin{aligned} \dot{x}_1(t) &= 2x_1(t) + x_2(t), \\ 0 &= x_1(t) + x_2(t-h), \end{aligned}$$

is of advanced type, as its characteristic function is

$$f(s) = \det H(s) = -(s-2)e^{-sh} - 1.$$

Also, we can express $x_2(t) = -x_1(t+h)$, and see that the system can be reduced to the advanced type equation $\dot{x}_1(t) = 2x_1(t) - x_1(t+h)$.

To avoid such "strange" properties the following class of systems was introduced in [4].

Definition 2. *System* (1) *is strangeness-free*, if there exists an invertible matrix

$$T_1 = \begin{pmatrix} R \\ P_1 \\ P_2 \end{pmatrix} \in \mathbf{R}^{n \times n},$$

where R, P_1 and P_2 are some blocks of rows, such that $P_1 E = 0$, $P_2(E\ A_0) = 0$, matrix

$$S_1 = \begin{pmatrix} RE \\ P_1 A_0 \\ P_2 A_1 \end{pmatrix}$$

is invertible, and, in addition, $P_2 B = 0$.

Remark. *In contrast to the definition in [4], we put an additional condition* $P_2B = 0$, *as we consider systems with input.*

It is easy to show that strangeness-free system has a unique solution for any consistent initial function. Another important property of such systems is in the following theorem.

Theorem 1 ([4]). *Strangeness-free system* (1) *is exponentially stable if and only if the spectrum* Λ *is located in the open left half-plane and is separated from the imaginary axis.*

If system (1) is strangeness-free, premultiplying by T_1 leads to the "stepped" form

$$\begin{array}{rcccccc} \frac{d}{dt}\big(REx(t)\big) & = & RA_0x(t) & + & RA_1x(t-h) & + & RBu(t), \\ 0 & = & P_1A_0x(t) & + & P_1A_1x(t-h) & + & P_1Bu(t), \\ 0 & = & & & P_2A_1x(t-h). & & \end{array} \tag{2}$$

The system is divided into three parts. The first part is the set of differential-difference equations, the second part is the set of difference equations, and the third part consists of algebraic equations. If P_2B would be different from zero, the algebraic part of the system would have the form

$$P_2A_1x(t-h) = -P_2Bu(t),$$

and would violate the causality principle. Thus, it is reasonable to assume that the algebraic part is not affected by the input.

It is important to point out that the "stepped" form is not obtained using a coordinate transformation, i. e., system (2) is still expressed in terms of the original variable x.

The following lemma characterizes the dimensions of the row blocks R, P_1, P_2 in Definition 2.

Lemma 1. *If system* (1) *is strangeness-free, then matrix* T_1 *in Definition 2 must satisfy,*

$$R \in \mathbf{R}^{r_1\times n}, \quad P_1 \in \mathbf{R}^{(r_2-r_1)\times n}, \quad P_2 \in \mathbf{R}^{(n-r_2)\times n},$$

where $r_1 = \operatorname{rank} E$, $r_2 = \operatorname{rank}(E\ A_0)$.

Proof. The invertibility of T_1 and S_1 implies that R, P_1, P_2, RE, P_1A_0, P_2A_1 are of full row rank. First, obviously, the rows of matrices P_1 and P_2 together form a basis for the left null space of matrix E. The dimension of this space is $n - r_1$. Therefore, $R \in \mathbf{R}^{r_1\times n}$. Second,

$$r_2 = \operatorname{rank}(E\ A_0) = \operatorname{rank} T_1(E\ A_0) = \operatorname{rank}\begin{pmatrix} RE & RA_0 \\ 0 & P_1A_0 \\ 0 & 0 \end{pmatrix} = r_1 + \tilde{r},$$

where $\tilde{r}$ is the numbers of rows in P_1. Hence, $\tilde{r} = r_2 - r_2$, and $P_1 \in \mathbf{R}^{(r_2-r_1)\times n}$. □

In the next section, we show how to reduce a strangeness-free system to a system of a better investigated subclass of systems, introduced in the following definition.

Definition 3. The *non-delayed part of system* (1) *is strangeness-free*, if there exists an invertible matrix

$$T_2 = \begin{pmatrix} R \\ P \end{pmatrix} \in \mathbf{R}^{n\times n},$$

such that $PE = 0$, and matrix

$$S_2 = \begin{pmatrix} RE \\ PA_0 \end{pmatrix}$$

is invertible.

It is easy to show that in this definition we must have $R \in \mathbf{R}^{r_1 \times n}$, $P \in \mathbf{R}^{(n-r_1)\times n}$, where $r_1 = \operatorname{rank} E$.

Remark. *Equivalently, we can say that the non-delayed part of system* (1) *is strangeness-free, if the system is strangeness-free for* $A_1 = 0$*, i. e., this concept does not take into account the delayed term of the system.*

The following lemma shows the equivalence between the concept of a system with strangeness-free non-delayed part and the assumption that the system is regular and impulse-free, which is rather standard in the theory of delay-free control systems. The proof can be found in [5].

Lemma 2. *The following statements are equivalent:*

1. The non-delayed part of system (1) *is strangeness-free.*

2. There exist two nonsingular matrices F_1*,* $F_2 \in \mathbf{R}^{n\times n}$*, such that*

$$F_1 E F_2 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad F_1 A_0 F_2 = \begin{pmatrix} A_0^{(1)} & A_0^{(2)} \\ 0 & I \end{pmatrix}. \tag{3}$$

3. The pair $(E\ A_0)$ *is regular and impulse-free, i. e., there exist two nonsingular matrices* F_3*,* $F_4 \in \mathbf{R}^{n\times n}$*, such that*

$$F_3 E F_4 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad F_3 A_0 F_4 = \begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix},$$

where J *is a Jordan block matrix.*

The dimension of the first blocks of all these matrices is $r_1 \times r_1$*, where* $r_1 = \operatorname{rank} E$*. The other blocks are of appropriate dimension.*

Item 3 of the lemma is connected to the Weierstraß canonical form for the deferential-algebraic delay-free systems; see, e. g., [16].

Corollary 1. *In Definition 3, one can replace* E *and* A_0 *by* E^T *and* A_0^T*, respectively, to obtain an equivalent definition.*

Remark. *By Item 2 of Lemma 2, the system of coupled differential-difference equations*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + By(t-h), \\ y(t) &= Cx(t) + Dy(t-h), \end{aligned}$$

is a particular case of the system with strangeness-free non-delayed part. Indeed, the substitution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \begin{pmatrix} \tilde{x}(t) \\ \tilde{y}(t) \end{pmatrix}$$

reduces the matrices of the system to the form (3)*.*

In what follows we will use the fact that the concepts from Definitions 2, 3 are invariant with respect to the choice of T_1 and T_2, respectively. This is specified in the following lemma.

Lemma 3. *Let* $r_1 = \operatorname{rank} E$*,* $r_2 = \operatorname{rank}(E\ A_0)$*. System* (1) *is strangeness free if and only if for any invertible matrix*

$$T_1 = \begin{pmatrix} R \\ P_1 \\ P_2 \end{pmatrix} \in \mathbf{R}^{n\times n},$$

such that $P_1 \in \mathbf{R}^{(r_2-r_1)\times n}$*,* $P_1 E = 0$*,* $P_2 \in \mathbf{R}^{(n-r_2)\times n}$*, and* $P_2(E\ A_0) = 0$*, matrix*

$$S_1 = \begin{pmatrix} RE \\ P_1 A_0 \\ P_2 A_1 \end{pmatrix}$$

is invertible, and, in addition, $P_2B = 0$.

The non-delayed part of system (1) *is strangeness-free if and only if for any invertible matrix*

$$T_2 = \begin{pmatrix} R \\ P \end{pmatrix} \in \mathbf{R}^{n\times n},$$

such that $P \in \mathbf{R}^{(n-r_1)\times n}$, *and* $PE = 0$, *matrix*

$$S_2 = \begin{pmatrix} RE \\ PA_0 \end{pmatrix}$$

is invertible.

Proof. We prove the first part of the lemma, the second part can be proven similarly. The sufficiency holds true by definition. Therefore, we restrict ourselves to proving the necessity. By the definition of strangeness-free systems, there exists invertible T_1, such that S_1 is also invertible, and $P_2B = 0$. We take another arbitrary invertible matrix

$$\bar{T}_1 = \begin{pmatrix} \bar{R} \\ \bar{P}_1 \\ \bar{P}_2 \end{pmatrix},$$

such that $\bar{P}_1E = 0$, $\bar{P}_2(E\ A_0) = 0$ with the dimension of the blocks, as specified in the statement of this lemma, and we show that the corresponding $\bar{S}_1$ is invertible, and $\bar{P}_2B = 0$. Obviously, there exists invertible matrix

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix},$$

such that $\bar{T}_1 = QT_1$. If we multiply this equality by E, we can show that $Q_{21} = 0$, $Q_{31} = 0$. If subsequently we multiply the equality by A_0, we can show that $Q_{32} = 0$, as matrices RE and P_1A_0 are of full rank. Therefore,

$$\bar{R} = Q_{11}R + Q_{12}P_1 + Q_{13}P_2, \quad \bar{P}_1 = Q_{22}P_1 + Q_{23}P_2, \quad \bar{P}_2 = Q_{33}P_2,$$

and matrices Q_{11}, Q_{22}, Q_{33} are invertible. Now it is easy to see that

$$\bar{S}_1 = \begin{pmatrix} \bar{R}E \\ \bar{P}_1A_0 \\ \bar{P}_2A_1 \end{pmatrix} = \begin{pmatrix} Q_{11}RE \\ Q_{22}P_1A_0 \\ Q_{33}P_2A_1 \end{pmatrix}$$

is invertible, and $\bar{P}_2B = Q_{33}P_2B = 0$. □

In the next section, we present an approach that allows to reduce strangeness-free system to the system with strangeness-free non-delayed part, which preserves some important properties. The approach does not require the computation of matrix T_2 from Definitions 3, it only requires computation of its P-part.

3 Reducing strangeness-free system to the system with strangeness-free non-delayed part

We show that in most cases there is no necessity to consider generic strangeness-free systems, as they can be easily reduced to the systems with strangeness-free non-delayed part.

First, we need to compute the number $r_2 = \operatorname{rank}(E\ A_0)$. If $r_2 = n$, the following two situations are possible:

1) System (1) already has a strangeness-free non-delayed part, and we do not need to transform it. In the next section we will show how to check this property.
2) The system is not strangeness-free. This case is out of the scope of our research.

Thus, we can assume that $r_2 < m$.

Assumption 1. *$r_2 = \operatorname{rank}(E\ A_0) < n$.*

In order to introduce the system with strangeness-free non-delayed part, corresponding to the strangeness-free system (1), we need matrices $\widetilde{P}_2$, $\widetilde{X}_2$, which result from the application of the following simple algorithm.

Algorithm 1.

I. *As $r_2 < n$, we can compute $\widetilde{P}_2 \in \mathbf{R}^{(n-r_2)\times n}$ of full rank, such that*

$$\widetilde{P}_2(E\ A_0) = 0.$$

Check the condition $\widetilde{P}_2 B = 0$. If this condition is violated, we conclude that the system is not strangeness-free, otherwise, go to the next step.

II. *Compute $\widetilde{X}_2 \in \mathbf{R}^{n\times(n-r_2)}$, such that*

$$\widetilde{P}_2\widetilde{X}_2 = I.$$

Construct now the system

$$\frac{d}{dt}\big(Ex(t)\big) = A_0x(t) + A_1x(t-h) + \widetilde{X}_2\widetilde{P}_2A_1\big(x(t) - x(t-h)\big) + Bu(t), \tag{4}$$

which has the following properties.

Theorem 2. *Let Assumption 1 holds true.*

1. Systems (1) and (4) have the same spectrum.

2. System (1) is strangeness-free if and only if system (4) has strangeness-free non-delayed part.

3. If system (1) is strangeness-free, the transfer matrix for this system is equal to the transfer matrix for system (4) with $y(t) = Cx(t)$.

Proof. Item 1. The characteristic matrix for system (4) is

$$\begin{aligned}\widetilde{H}(s) &= sE - A_0 - e^{-sh}A_1 - \widetilde{X}_2\widetilde{P}_2A_1(1 - e^{-sh}) \\ &= H(s) - (e^{sh} - 1)\widetilde{X}_2\widetilde{P}_2\left(sE - A_0 - H(s)\right).\end{aligned}$$

As $\widetilde{P}_2(E\ A_0) = 0$ by construction, we get

$$\widetilde{H}(s) = (I + (e^{sh} - 1)\widetilde{X}_2\widetilde{P}_2)H(s).$$

We can compute the determinant of the first factor by Schur's formulas, taking into account that $\widetilde{P}_2\widetilde{X}_2 = I$:

$$\det(I + (e^{sh} - 1)\widetilde{X}_2\widetilde{P}_2) = \det(I + (e^{sh} - 1)\widetilde{P}_2\widetilde{X}_2) = \det(e^{sh}I) = e^{sh(n-r_2)} \neq 0.$$

Thus, $\det H(s) = 0$ if and only if $\det \widetilde{H}(s) = 0$.

Item 2. The non-delayed part of system (4) is

$$\frac{d}{dt}\big(Ez(t)\big) = \widetilde{A}_0z(t), \tag{5}$$

where $\widetilde{A}_0 = A_0 + \widetilde{X}_2\widetilde{P}_2A_1$. Since system (1) is strangeness-free, there exists matrix T_1 from Definition 2, where block P_2 can be chosen equal to $\widetilde{P}_2$ (by Lemma 3). Note that $P_1\widetilde{X}_2 = 0$. Indeed, otherwise rows of P_1 and $\widetilde{P}_2$ are not linearly independent, and the matrix T_1 is not invertible. If we denote the second part of matrix T_1 by P,

$$P = \begin{pmatrix} P_1 \\ \widetilde{P}_2 \end{pmatrix},$$

matrix

$$T_2 = \begin{pmatrix} R \\ P \end{pmatrix}$$

satisfies Definition 3 for system (5), because $PE = 0$, and

$$S_2 = \begin{pmatrix} RE \\ P\widetilde{A}_0 \end{pmatrix} = \begin{pmatrix} RE \\ P_1 A_0 \\ \widetilde{P}_2 \widetilde{X}_2 \widetilde{P}_2 A_1 \end{pmatrix} = \begin{pmatrix} RE \\ P_1 A_0 \\ \widetilde{P}_2 A_1 \end{pmatrix}$$

is invertible.

Assume now that system (4) has strangeness-free non-delayed part. There exists

$$T_2 = \begin{pmatrix} R \\ P \end{pmatrix}$$

satisfying Definition 3. One can find $P_0 \in \mathbf{R}^{(n-r_1)\times(n-r_1)}$, such that the second part of T_2 can be expressed as

$$P = P_0 \begin{pmatrix} P_1 \\ \widetilde{P}_2 \end{pmatrix},$$

where $\widetilde{P}_2$ is the matrix that has been defined in Algorithm 1. Obviously, matrix

$$T_1 = \begin{pmatrix} R \\ P_1 \\ \widetilde{P}_2 \end{pmatrix}$$

satisfies Definition 2, and system (1) is strangeness-free.

Item 3. Note, first, that

$$\left(I + (e^{sh} - 1)\widetilde{X}_2 \widetilde{P}_2\right)^{-1} B = B.$$

As has been shown in the proof of Item 1,

$$\widetilde{H}(s) = (I + (e^{sh} - 1)\widetilde{X}_2 \widetilde{P}_2) H(s).$$

Thus, the transfer matrix for system (4)

$$\widetilde{G}(s) = C\widetilde{H}^{-1}(s)B = CH^{-1}(s)\left(I + (e^{sh} - 1)\widetilde{X}_2 \widetilde{P}_2\right)^{-1} B = G(s).$$

□

The basic idea behind the construction of system (4) can be easily explained by mean of the "stepped" form (2). We make time shift in the algebraic part of the system (such transformation does not change the spectrum), and come back, premultiplying by T_1^{-1}. In our approach transformation T_1 is hidden, and we just need the last part of the transformation – matrix P_2. In fact, matrix $\widetilde{P}_2$ is the third part of T_1 (i. e., matrix P_2), but we use different notation to indicate that $\widetilde{P}_2$ can be computed independently of T_1, i. e., without computation of T_1.

As a result, the only difference between (1) and (4) is an additional term on the right hand side, which is of low rank, as $\widetilde{X}_2 \widetilde{P}_2$ has rank $n - r_1$.

As shown in Theorem 2, systems (1) and (4) are equivalent in the frequency-domain, but in the time-domain these systems are not equivalent. The following example illustrates this fact.

Example 3. *We compare system*

$$\begin{aligned} \dot{x}_1(t) &= x_2(t-1), \\ 0 &= x_2(t-1), \end{aligned} \tag{6}$$

with the system, obtained by time-shift in the second equation:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t-1), \\ 0 &= x_2(t). \end{aligned} \tag{7}$$

It is easy to note that the former system is strangeness-free, whereas the latter system is with strangeness-free non-delayed part. It is easy to check that these systems have the same spectrum $\Lambda = \{0\}$. But the general solution of system (6) *for $t \geqslant 0$ has the form*

$$x(t) = \begin{pmatrix} C \\ 0 \end{pmatrix},$$

where C is an arbitrary constant, whereas the general solution of system (7) *for $t \geqslant 0$ has the form*

$$x(t) = \begin{pmatrix} \tilde{\varphi}(t) \\ 0 \end{pmatrix},$$

where $\tilde{\varphi}(t)$ is an absolutely continuous function, which is constant for $t \geqslant 1$. Thus, the set of solutions for system (7) *is broader than the set for system* (6), *but every solution of system* (7) *can be presented as a sum of a solution of system* (6) *and the so-called small solution (see, [3]), which is equal to zero, starting from $t = 1$.*

4 Reducing system with strangeness-free non-delayed part to the neutral type non-singular system

In this section, we consider a system of the form (1) with the strangeness-free non-delayed part.

4.1 Characterization of systems with strangeness-free non-delayed part

First, we need to compute the number $r_1 = \operatorname{rank} E$. If $r_1 = n$, the system is non-singular, and we do not need to reduce it to neutral type system. We will assume that $r_1 < n$.

Assumption 2. $r_1 = \operatorname{rank} E < n$.

We start with another simple algorithm, similar to Algorithm 1.

Algorithm 2.

I. *As $r_1 < n$, we can compute $\widetilde{P} \in \mathbf{R}^{(n-r_1)\times n}$ of full rank, such that*

$$\widetilde{P}E = 0.$$

II. *Compute $\widetilde{X} \in \mathbf{R}^{n\times(n-r_1)}$, such that*

$$\widetilde{P}\widetilde{X} = M^{-1},$$

where M is an arbitrary fixed real invertible matrix of dimension $(n - r_1) \times (n - r_1)$. As will be shown later, it is better choosing a Hurwitz matrix, relevant in the context of preservation stability.

In terms of $\widetilde{P}$ and $\widetilde{X}$ we derive a necessary and sufficient condition to check, weather the non-delayed part of the system is strangeness-free or not, without transformation of the system.

Theorem 3. *Let Assumption 2 holds true. The non-delayed part of system* (1) *is strangeness-free if and only if*

$$\operatorname{rank}(E + \widetilde{X}\widetilde{P}A_0) = n.$$

Proof. As rank $\widetilde{X} = n - r_1$, there exists a full rank matrix $R \in \mathbf{R}^{r_1 \times n}$, such that $R\widetilde{X} = 0$. It is easy to see that rows of R are linearly independent with rows of $\widetilde{P}$, i. e.,

$$T_2 = \begin{pmatrix} R \\ \widetilde{P} \end{pmatrix}$$

is invertible. Matrix

$$S_2 = \begin{pmatrix} RE \\ \widetilde{P}A_0 \end{pmatrix}$$

is invertible if and only if $\operatorname{rank}(E + \widetilde{X}\widetilde{P}A_0) = n$. Indeed,

$$\begin{aligned} \operatorname{rank}(E + \widetilde{X}\widetilde{P}A_0) &= \operatorname{rank} T_2(E + \widetilde{X}\widetilde{P}A_0) \\ &= \operatorname{rank} \begin{pmatrix} RE \\ \widetilde{P}\widetilde{X}\widetilde{P}A_0 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} I & 0 \\ 0 & M^{-1} \end{pmatrix} \begin{pmatrix} RE \\ \widetilde{P}A_0 \end{pmatrix}. \end{aligned}$$

By Lemma 3, the invertibility of S_2 is equivalent to the strangeness-free property of the non-delayed part of system (1). □

4.2 Reducing to the neutral type delay systems

Using the introduced auxiliary matrices $\widetilde{P}$ and $\widetilde{X}$, for system (1) we can construct neutral type system with a prescribed additional dynamics, determined by the chosen matrix M:

$$\frac{d}{dt}\Big((E + \widetilde{X}\widetilde{P}A_0)x(t) + \widetilde{X}\widetilde{P}A_1 x(t-h)\Big) = A_0 x(t) + A_1 x(t-h) + Bu(t) - \widetilde{X}\widetilde{P}B\frac{d}{dt}u(t). \tag{8}$$

Theorem 4. *Let Assumption 2 holds true.*

1. System (1) *has strangeness-free non-delayed part if and only if system* (8) *is a non-singular system of neutral type.*

2. The spectrum of system (8) *is equal to*

$$\Lambda \cup \sigma(M),$$

where $\sigma(M)$ is the spectrum of matrix M, and Λ is the spectrum of system (1)*.*

3. If system (1) *has strangeness-free non-delayed part, the transfer matrix for this system is equal to the transfer matrix for system* (8) *with $y(t) = Cx(t)$.*

Proof. Item 1 is a trivial consequence of Theorem 3.

Item 2. Let

$$H_1(s) = s(E + \widetilde{X}\widetilde{P}A_0) + s\widetilde{X}\widetilde{P}A_1 e^{-sh} - A_0 - A_1 e^{-sh}$$

be the characteristic matrix for system (8). It is easy to see that

$$\begin{aligned} H_1(s) &= H(s) + s\widetilde{X}\widetilde{P}(A_0 + A_1 e^{-sh}) = H(s) + s\widetilde{X}\widetilde{P}(sE - H(s)) \\ &= (I - s\widetilde{X}\widetilde{P})H(s) + s^2\widetilde{X}\widetilde{P}E = (I - s\widetilde{X}\widetilde{P})H(s). \end{aligned} \tag{9}$$

An application of Schur's formulas completes the proof:

$$\det(I - s\widetilde{X}\widetilde{P}) = \det(I - s\widetilde{P}\widetilde{X}) = \det(I - sM^{-1}) = \det(-M^{-1})\det(sI - M).$$

Item 3. The transfer matrix for system (8) is

$$G_1(s) = CH_1^{-1}(s)\left(B - s\widetilde{X}\widetilde{P}B\right) = CH^{-1}(s)\left(I - s\widetilde{X}\widetilde{P}\right)^{-1}\left(I - s\widetilde{X}\widetilde{P}\right)B = G(s).$$

□

Eigenvalues of the matrix M make no influence on the transfer matrix, as they are not spectrally controllable. This is a consequence of the equality for the following augmented matrix

$$\begin{pmatrix} H_1(s) & B - s\widetilde{X}\widetilde{P}B \end{pmatrix} = \begin{pmatrix} I - s\widetilde{X}\widetilde{P} \end{pmatrix} \begin{pmatrix} H(s) & B \end{pmatrix}.$$

Let us now give an alternative, more intuitive explanation, based on the "stepped" form (2), how system (8) has been obtained. For the sake of simplicity, we consider the case $B = 0$. If we premultiply system (1) with strangeness-free non-delayed part by T_2 from Definition 3, we can separate the difference part from the differential-difference one:

$$\begin{aligned} \frac{d}{dt}\left(REx(t)\right) &= RA_0x(t) + RA_1x(t-h), \\ 0 &= PA_0x(t) + PA_1x(t-h). \end{aligned}$$

Now, applying operator

$$M^{-1}\frac{d}{dt} - I$$

to the difference part of the system, we arrive at the system

$$\begin{aligned} \frac{d}{dt}\left(REx(t)\right) &= RA_0x(t) + RA_1x(t-h), \\ \frac{d}{dt}\left(M^{-1}PA_0x(t) + M^{-1}PA_1x(t-h)\right) &= PA_0x(t) + PA_1x(t-h) \end{aligned}$$

with the spectrum $\Lambda \cup \sigma(M)$. If we now premultiply this system by

$$T_2^{-1} = \begin{pmatrix} X_0 & X_1 \end{pmatrix},$$

we obtain (8) with $\widetilde{X} = X_1M^{-1}$. Note that in (8) transformation T_2 is hidden, and we use only the second part – matrix P.

4.3 Implication for the stability analysis

Theorems 2, 3, 4 can be applied to the stability analysis of system (1). We consider a system of the form (1). First, we can construct system (4). If the system is with strangeness-free non-delayed part, then we can conclude that the original system is strangeness-free by Theorem 2. In the second step, we construct system (8). To this end we choose Hurwitz matrix M in Algorithm 2 to guarantee that the additional spectrum of system (8) is located in the left half of the complex plane. For the sake of simplicity, we can take, for instance, $M = \alpha I$, where $\alpha < 0$.

As the additional dynamics are stable, systems (1) and (8) are equivalent in the sense of exponential stability. We can apply an existing method for the stability analysis of the neutral type system (8) (see, e. g., [10, 19, 15, 18, 7]) to define the stability of (1).

As has been shown in [1], the spectrum of neutral type system (8) consists of a finite set of eigenvalues, and of a finite number of countable chains of eigenvalues. Some of these chains may be of retarded type that means that the real part of eigenvalues tends to $-\infty$, and some of them may be of neutral type. The neutral type chains lie along the vertical lines of the form $\mathrm{Re}(s) = \ln|z|/h$, where z is a non-zero eigenvalue of the matrix

$$D = (E + \widetilde{X}\widetilde{P}A_0)^{-1}\widetilde{X}\widetilde{P}A_1.$$

Note that, as $\widetilde{X}$ and $\widetilde{P}$ are not square matrices, some of the eigenvalues of D are zero, and non-zero eigenvalues coincide with the non-zero eigenvalues of

$$\widetilde{D} = \widetilde{P}A_1(E + \widetilde{X}\widetilde{P}A_0)^{-1}\widetilde{X}.$$

In particular, system (1) is of retarded type (has no neutral type chains of eigenvalues), if $\widetilde{P}A_1 = 0$.

The following lemma, based on the classical Lyapunov result for discrete systems, is well known for the neutral type systems.

Lemma 4. *Let system* (1) *have strangeness-free non-delayed part. A necessary stability condition for system* (1) *is that all eigenvalues of D (equivalently, $\widetilde{D}$) lie in the open unit circle, or equivalently, the unique solution U of equation*

$$U - D^T U D = W$$
$$\left(or \quad U - \widetilde{D}^T U \widetilde{D} = W \right),$$

where W is an arbitrary positive definite matrix of appropriate dimension, is positive definite.

Remark. *By Corollary 1, all the results of this section can be formulated in terms of "right" auxiliary elements. We could take $\widetilde{P}_1 \in \mathbf{R}^{n\times(n-r_1)}$ as a basis of the right null space of E, i. e., as a full rank matrix, such that $E\widetilde{P}_1 = 0$. In this case we need to choose $\widetilde{X}_1 \in \mathbf{R}^{(n-r_1)\times n}$, such that $\widetilde{X}_1\widetilde{P}_1 = M^{-1}$. The corresponding neutral type system is*

$$\frac{d}{dt}\left((E + A_0\widetilde{P}_1\widetilde{X}_1)x(t) + A_1\widetilde{P}_1\widetilde{X}_1 x(t-h)\right) = A_0 x(t) + A_1 x(t-h) + Bu(t) \tag{10}$$

with

$$y(t) = Cx(t) - C\widetilde{P}_1\widetilde{X}_1\frac{d}{dt}x(t).$$

It is easy to prove that $E + A_0\widetilde{P}_1\widetilde{X}_1$ is invertible if and only if the non-delayed part of system (1) *is strangeness-free, the neutral type chains are defined by*

$$D = A_1\widetilde{P}_1\widetilde{X}_1(E + A_0\widetilde{P}_1\widetilde{X}_1)^{-1},$$

or equivalently, by

$$\widetilde{D} = \widetilde{X}_1(E + A_0\widetilde{P}_1\widetilde{X}_1)^{-1}A_1\widetilde{P}_1.$$

Also, one can estimate the decay rate for the solutions of system (1), using the results for neutral type systems; see, e. g., [9, 13, 12, 18, 2]. Obviously, it is important to choose the additional spectrum to be located sufficiently far to the left of the imaginary axis of the complex plane.

5 The $\mathcal{H}_2$ norm characterization

It has been shown that every strangeness-free system can be reduced to the system with strangeness-free non-delayed part with the same transfer matrix. Therefore, in this section we can assume that system (1) has strangeness-free non-delayed part. Also we assume that the system is exponentially stable.

The $\mathcal{H}_2$ norm of the transfer matrix $G(s) = CH^{-1}(s)B$ for system (1) is

$$\|G\|_{\mathcal{H}_2} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Tr}\left(G^*(i\omega)G(i\omega)\right)\, d\omega}. \tag{11}$$

In contrast to systems with invertible matrix E, the $\mathcal{H}_2$ norm of $G(s)$ is either finite or infinite.

Example 4. *We consider a simple exponentially stable SISO system of the form* (1) *with matrices:*

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & -1/2 \\ 1 & -1/2 \end{pmatrix},$$
$$B = \begin{pmatrix} 1 \\ b_2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & c_2 \end{pmatrix},$$

where b_2, c_2 are some parameters, $h = 1$. The transfer matrix has the form

$$G(s) = \frac{2b_2c_2s + 2(1 + b_2c_2) + (1 - b_2 + 2c_2 - 2b_2c_2)e^{-s}}{s(2 + e^{-s}) + (2 - e^{-s})}.$$

Obviously, the $\mathcal{H}_2$ norm of G is finite if and only if $b_2c_2 = 0$, i. e., if $b_2 = 0$ or $c_2 = 0$. But in both cases the transfer matrix is still dependent on the value of the second parameter:

$$G(s) = \frac{2 + (1 - b_2 + 2c_2)e^{-s}}{s(2 + e^{-s}) + (2 - e^{-s})}.$$

In the next subsection, we give a criterion for finiteness of the norm of the transfer matrix.

5.1 Finiteness of the $\mathcal{H}_2$ norm

As the non-delayed part of system (1) is assumed to be strangeness-free, we can introduce matrix

$$K = (E + \widetilde{X}\widetilde{P}A_0)^{-1},$$

where $\widetilde{X}$, $\widetilde{P}$ are from Algorithm 2. In this case matrices D and $\widetilde{D}$ from Subsection 4.3 take the form

$$D = K\widetilde{X}\widetilde{P}A_1, \quad \widetilde{D} = \widetilde{P}A_1K\widetilde{X}.$$

Now we can give a necessary and sufficient condition for the finiteness of the $\mathcal{H}_2$ norm of the transfer matrix for system (1).

Theorem 5. *Let system* (1) *have strangeness-free non-delayed part and be exponentially stable. The following statements are equivalent:*

1. *The $\mathcal{H}_2$ norm of the transfer matrix for system* (1) *is finite.*
2. $CK\widetilde{X}(I + e^{-sh}\widetilde{D})^{-1}\widetilde{P}B \equiv 0$ *for all* $s \in \mathbf{C}$ *such that* $\mathrm{Re}(s) \geqslant 0$.
3. $CK\widetilde{X}\widetilde{D}^k\widetilde{P}B = 0$, $k = \overline{0, n - r_1 - 1}$, *where* $r_1 = \mathrm{rank}\, E$.
4. $U\widetilde{P}B = 0$, *where U is the unique solution of the discrete Lyapunov equation*

$$U - \widetilde{D}^TU\widetilde{D} = W \quad \text{with } W = (CK\widetilde{X})^TCK\widetilde{X}.$$

Proof. Equivalence between Item 1 and Item 2. By Woodbury formula,

$$(I + e^{-sh}\widetilde{D})^{-1} = I - e^{-sh}\widetilde{P}A_1(I + e^{-sh}D)^{-1}K\widetilde{X}.$$

It is easy now to check that the transfer matrix can be decomposed as follows:

$$G(s) = G^{(1)}(s) - G^{(2)}(s),$$

where

$$G^{(1)}(s) = C(I + e^{-sh}D)^{-1}KEH^{-1}(s)B,$$
$$G^{(2)}(s) = CK\widetilde{X}(I + e^{-sh}\widetilde{D})^{-1}\widetilde{P}B.$$

We now prove that $\|G^{(1)}\|_{\mathcal{H}_2}$ is a finite number. As $\widehat{H}(s, z) = sE - A_0 - zA_1$ is a rational function of two variables s and z, and by formula (11), it is enough to show that $EH^{-1}(i\omega)$, tends to zero, as $w \to \infty$, $\omega \in \mathbf{R}$. This is an evident corollary of the following equality:

$$EH^{-1}(s) = \frac{1}{s}I + \frac{1}{s}\left(A_0 + e^{-sh}A_1\right)H^{-1}(s).$$

Thus, we have proven that $\|G^{(1)}\|_{\mathcal{H}_2}$ is finite. By the triangle inequality,

$$\|G^{(2)}\|_{\mathcal{H}_2} - \|G^{(1)}\|_{\mathcal{H}_2} \leqslant \|G\|_{\mathcal{H}_2} \leqslant \|G^{(1)}\|_{\mathcal{H}_2} + \|G^{(2)}\|_{\mathcal{H}_2},$$

the finiteness of $\|G\|_{\mathcal{H}_2}$ is equivalent to the finiteness of $\|G^{(2)}\|_{\mathcal{H}_2}$, which is finite if and only if $G^{(2)}$ is zero.

Equivalence between Item 2 and Item 3. This simple fact follows from the series

$$(I + e^{-sh}\widetilde{D})^{-1} = \sum_{j=0}^{\infty}(-1)^j e^{-jsh}\widetilde{D}^j$$

and Cayley–Hamilton theorem.

Equivalence between Item 3 and Item 4. Item 3 means that all the positive semidefinite matrices

$$B^T\widetilde{P}^T(\widetilde{D}^T)^k W\widetilde{D}^k\widetilde{P}B, \quad k = 0, 1, 2, \dots,$$

are zero, which is equivalent to the equality

$$B^T\widetilde{P}^T\left(\sum_{k=0}^{\infty}(\widetilde{D}^T)^k W\widetilde{D}^k\right)\widetilde{P}B = 0.$$

The matrix in the brackets satisfies the discrete Lyapunov equation. And this is the unique solution by Lemma 4. □

Corollary 2. *We can mark two important particular cases. The $\mathcal{H}_2$ norm of the transfer matrix for exponentially stable system* (1) *with strangeness-free non-delayed part is finite, if one of the following statements holds:*

1. $\widetilde{P}B = 0$ (that is equivalent to the existence of B_0, such that $B = EB_0$), i.e., the difference part of the system is not affected by the input.

2. $CK\widetilde{X} = 0$ (that is equivalent to the existence of C_0, such that $C = C_0E$), i.e., components of the state vector x, corresponding to the difference part, are not measurable (shortly, the difference part of the system is not measurable).

Proof. The only thing which is not obvious in this corollary is the equivalence between $CK\widetilde{X} = 0$ and the existence of C_0, such that $C = C_0E$. As $\widetilde{X}$ is of full rank $n - r_1$, the left null space of the matrix $K\widetilde{X}$ is of the dimension r_1. The row space of E is of the same dimension. Thus, it remains to deduce the equality $EK\widetilde{X} = 0$ to prove that the left null space of $K\widetilde{X}$ coincides with the row space of E.

Like in the proof of Theorem 3, we can construct an invertible matrix

$$T_2 = \begin{pmatrix} R \\ \widetilde{P} \end{pmatrix},$$

where R is such that $R\widetilde{X} = 0$. Hence,

$$\begin{aligned} EK\widetilde{X} &= T_2^{-1}\begin{pmatrix} RE \\ 0 \end{pmatrix}\left(T_2^{-1}\begin{pmatrix} RE \\ \widetilde{P}\widetilde{X}\widetilde{P}A_0 \end{pmatrix}\right)^{-1} T_2^{-1}\begin{pmatrix} 0 \\ \widetilde{P}\widetilde{X} \end{pmatrix} \\ &= T_2^{-1}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} RE \\ \widetilde{P}\widetilde{X}\widetilde{P}A_0 \end{pmatrix}\begin{pmatrix} RE \\ \widetilde{P}\widetilde{X}\widetilde{P}A_0 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ \widetilde{P}\widetilde{X} \end{pmatrix} = 0. \end{aligned}$$

□

It is easy to see that Items 1 and 2 are equivalent to the conditions

$$\operatorname{rank} E = \operatorname{rank}(E\ B), \quad \operatorname{rank} E = \operatorname{rank}\begin{pmatrix} E \\ C \end{pmatrix},$$

respectively.

The two conditions in Corollary 2 correspond to $b_2 = 0$, respectively $c_2 = 0$, in Example 4. But it is worth mentioning that in general two options from Corollary 2 do not exhaust all the possibilities, where the $\mathcal{H}_2$ norm is finite. In the next subsection, we outline a broader class of systems with finite $\mathcal{H}_2$ norm, but here we give an example.

Example 5. *We consider exponentially stable system* (1) *with the following matrices:*

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 3 \\ -3 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix},$$

$$C = \begin{pmatrix} -1 & 0 & -3 \end{pmatrix}.$$

Following Algorithm 2, we can take

$$\widetilde{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M = -I, \quad \widetilde{X} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we can compute

$$K = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \widetilde{D} = \frac{1}{2}\begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}.$$

One can verify that the third item of Theorem 5 holds, i. e., $CK\widetilde{X}\widetilde{P}B = 0$ *and* $CK\widetilde{X}\widetilde{D}\widetilde{P}B = 0$. *Therefore, the* $\mathcal{H}_2$ *norm is finite, whereas both the conditions of Corollary 2 are violated.*

The finiteness of the $\mathcal{H}_2$ *norm can be easily checked, if we compute the transfer matrix:*

$$G(s) = \frac{-4 - 14e^{-s} + 8e^{-2s}}{4s + 12 - 4e^{-s} - (s-5)e^{-2s} - 6e^{-3s}}.$$

5.2 Computation of the $\mathcal{H}_2$ norm

The main drawback of the constructed neutral system (8) is the summand $\widetilde{X}\widetilde{P}B\frac{d}{dt}u(t)$. The computation of the $\mathcal{H}_2$ norm of the transfer matrix for non-singular system with the derivative in the input is not a trivial problem, and the $\mathcal{H}_2$ norm still can be infinite. Here we present the condition, which guarantees that the summand $\widetilde{X}\widetilde{P}B\frac{d}{dt}u(t)$ can be eliminated.

We introduce the system

$$\frac{d}{dt}\Big((E + \widetilde{X}\widetilde{P}A_0)x(t) + \widetilde{X}\widetilde{P}A_1x(t-h)\Big) = A_0x(t) + A_1x(t-h) + Bu(t), \tag{12}$$

which coincides with (8) up to the last summand.

Theorem 6. *Let system* (1) *have strangeness-free non-delayed part and be exponentially stable. The transfer matrix for system* (1) *is equal to the one for system* (12) *with* $y(t) = Cx(t)$ *if and only if the transfer function of system*

$$\frac{d}{dt}\Big((E + \widetilde{X}\widetilde{P}A_0)x(t) + \widetilde{X}\widetilde{P}A_1x(t-h)\Big) = A_0x(t) + A_1x(t-h) + \widetilde{X}\widetilde{P}Bu(t) \tag{13}$$

with $y(t) = Cx(t)$ *is equal to zero.*

This is the case, if the first condition from Corollary 2 holds true, i. e., the difference part of the system is not affected by the input.

Proof. To prove the result we just need to express explicitly the transfer matrices

$$G(s) = CH^{-1}(s)B, \quad G_2(s) = CH_1^{-1}(s)B, \quad G_3(s) = CH_1^{-1}(s)\widetilde{X}\widetilde{P}B$$

for systems (1), (12), and (13), respectively. Using equality (9), one can obtain

$$G(s) - G_2(s) = -sG_3(s).$$

Now the desired result follows immediately from the fact that the $\mathcal{H}_2$ norm of the transfer function for any neutral type non-singular system (without any derivatives in the input) is finite. □

Thus, if the condition of Theorem 6 holds, one can apply the following formula from [11] to compute the $\mathcal{H}_2$ norm for system (1):

$$\|G\|_{\mathcal{H}_2} = \sqrt{\operatorname{Tr}\left(B^T U(0) B\right)}, \tag{14}$$

where $U(0)$ is the delay Lyapunov matrix, associated with $W = C^T C$, at zero, for the neutral type system (12). One can find semi-analytic and numerical procedures in [14, 15] and in [11], respectively, to compute $U(0)$.

Similarly, using the elements $\widetilde{P}_1$, $\widetilde{X}_1$, introduced in the remark in Subsection 4.3, corresponding to the right null space of the matrix E, it is easy to prove the following "dual" theorem.

Theorem 7. *Let system* (1) *have strangeness-free non-delayed part and be exponentially stable. The transfer matrix for system* (1) *is equal to the one for system* (10) *with* $y(t) = Cx(t)$ *if and only if the transfer function for system* (10) *with* $y(t) = C\widetilde{P}_1\widetilde{X}_1 x(t)$ *is equal to zero.*

This is the case, if the second condition from Corollary 2 holds true, i. e., the difference part of the system is not measurable.

6 Illustrative examples

Example 6. *We consider the exponentially stabile system* (1) *with* $h = 1$ *and the following matrices:*

$$E = \begin{pmatrix} 2 & 4 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 0 & 2 \\ -3 & 3 & 5 \\ -3 & 3 & 6 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & -2 & -2 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 2 & 5 & 1 \end{pmatrix}.$$

The rank of the matrix E *is equal to* 1, *i. e.,* $r_1 = 1$. *The rank of the matrix* $(E\ A_0)$ *is equal to* $r_2 = 2$. *The first step is to follow Algorithm 1 to construct system* (4). *Compute*

$$\widetilde{P}_2 = \begin{pmatrix} 1 & 2 & -2 \end{pmatrix}, \quad \widetilde{X}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, system (4) *takes the form*

$$\begin{pmatrix} 2 & 4 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{x}(t) = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 3 & 5 \\ -3 & 3 & 6 \end{pmatrix} x(t) + \begin{pmatrix} -2 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} x(t-1) + \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} u(t). \tag{15}$$

On the second step for $M = -I$ *we construct*

$$\widetilde{P} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{X} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix},$$

following Algorithm 2, applied to system (15)*. By Theorem 3, system* (15) *has strangeness-free non-delayed part. Therefore, by the second item of Theorem 2, the original system is strangeness-free. Now we can reduce system* (15) *to the non-singular system of NDDEs* (12) *with*

$$E + \widetilde{X}\widetilde{P}A_0 = \begin{pmatrix} 6 & -2 & -10 \\ -1 & -2 & 0 \\ 3 & -3 & -6 \end{pmatrix}, \quad \widetilde{X}\widetilde{P}A_1 = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

As $\widetilde{P}B = 0$*, by Theorems 2 and 6, the transfer matrix of this system of NDDEs coincides with the transfer matrix of the original one. By formula* (14)*, we find*

$$\|G\|_{\mathcal{H}_2} \approx 1.3631.$$

Note that this coincides with the result that can be approximately obtained directly by formula (11)*.*

Example 7. *Similarly, the system from [17],*

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \dot{x}(t) = -x(t) + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} x(t-h) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} u(t), \tag{16}$$

can be reduced to a neutral type one. Let $y(t) = x(t)$*. By Theorem 3, the system has strangeness-free non-delayed part and the corresponding non-singular system of the form* (12) *is*

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \dot{x}(t) = -x(t) + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} x(t-h) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} u(t),$$

which is of retarded type. This system is exponentially stable, if $h < 1.209$*. Take, for instance,* $h = 1$*. By Theorem 6 and formula* (14)*, the* $\mathcal{H}_2$ *norm for system* (16) *is*

$$\|G\|_{\mathcal{H}_2} \approx 1.7827.$$

7 Conclusion

It is shown how the task of analysis of a strangeness-free system of DDAEs can be reduced to the analysis of a neutral type non-singular system. The approach is applied to the stability analysis, to the estimation of the decay rate for the solutions, and to the characterizing of the $\mathcal{H}_2$ norm of the transfer matrix. In particular, an algebraic necessary and sufficient condition for the finiteness of the $\mathcal{H}_2$ norm is obtained.

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